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LOCALIZED EFFECT OF CLAMP OR SOCKET END CONNECTIONS ON HELICAL WIRES IN MULTILAYERED CABLES

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Abstract—A complete theory of helical thin elastic rods is applied to analyze the effect of constraint due to clamp or socket end connections on changes to the lay angles of the constituent wires in multilayered cables. While the effect on stresses in the wires is small, the end connection causes slipping to occur between the wires which could contribute to wear or fretting damage near the end connection.

INTRODUCTION

Loading a straight multilayered cable by axial force and twisting moment applied to its ends produces changes in the lay angles of the helical wires comprising the cable. The change in the lay angle, in each layer of wires, is uniform well away from the ends; but near the ends there is a transition region over which the effect of constraint of the end connections, typically clamps or sockets, extends. In some recent experimental and theoretical work, Utting and Jones (1987a,b) have made an initial attempt to account for the occurrence of this transition region. In the present paper, a detailed analysis of the transition region is presented, based on a complete theory of helical thin elastic rods developed recently by the author, Ramsey (1988). Although this theory is based on the general theory of rods due to Green and Laws (1966), it does not employ directors. The present theory is relatively simple, including just four generalized strains which describe extension, twisting and two components of bending. Excluded, in particular, are extensional strains in the normal cross-section and transverse shear deformation. These additional effects have been considered recently by Naghdi and Rubin (1989) in connection with a contact problem in beam theory. However, these effects are beyond the scope of the present investigation, and to treat the interwire contact forces in the transition region on a consistent approximate basis, it is necessary to neglect changes in wire diameters. In order to simplify the analysis further, interwire friction is also neglected.

SUMMARY OF A LINEAR THEORY OF HELICAL THIN ELASTIC RODS

The essential results from the rod theory presented by Ramsey (1988) are now summarized as they apply to thin rods of circular cross-section. The motion of points on the center-line in the rod is described by a position vector $\mathbf{r} = \mathbf{r}(S, t)$, where S is arc length along the center-line in the undeformed state, and t is time. The initial state at $t = 0$ is the undeformed reference state. Material cross-sections of the rod defined as normal cross-sections in the undeformed rod remain plane and normal as the rod deforms. The angular velocity of these material cross-sections is described by the vector $\mathbf{\Omega} = \mathbf{\Omega}(S, t)$. A right-handed orthonormal set of base vectors $\mathbf{a}_i = \mathbf{a}_i(S, t)$, where $i = 1, 2, 3$, is associated with the rod center-line. The vector $\mathbf{a}_3$ in this set is the tangent vector, i.e.

$$\mathbf{a}_3 = \frac{\partial \mathbf{r}}{\partial s}, \tag{1}$$

where $s = s(S, t)$ is current arc length along the center-line. The vectors $\mathbf{a}_1$, $\mathbf{a}_2$ rotate about the center-line with the material cross-sections of the rod. Hence,

$$\dot{\mathbf{a}}_i = \mathbf{\Omega} \times \mathbf{a}_i, \tag{2}$$

where the superposed dot denotes differentiation with respect to t, the material coordinate S being held constant. Differentiation of the base vectors $\mathbf{a}_i$ with respect to S is described by a skew-symmetric matrix $\kappa_{ij} = \kappa_{ij}(S, t)$, i.e.

$$\frac{\partial \mathbf{a}_i}{\partial S} = \kappa_{ij}\mathbf{a}_j, \tag{3}$$

where the repeated index implies summation. The initial values of κ_{ij} in the undeformed reference state are denoted by K_{ij}. The skew-symmetric matrix formed by the difference $(\kappa_{ij} - \mathrm{K}_{ij})$ measures bending and twisting of the rod. It is convenient to express this skew-symmetric matrix in terms of the kinematic variables $\omega_k = \omega_k(S, t)$ defined by

$$e_{ijk}\omega_k = \kappa_{ij} - \mathrm{K}_{ij}, \tag{4}$$

where e_{ijk} are the permutation symbols. It can be shown that

$$\frac{\partial \mathbf{\Omega}}{\partial S} = \dot{\omega}_k \mathbf{a}_k. \tag{5}$$

The extensional strain ε of the rod center-line, defined by

$$\frac{\partial s}{\partial S} = 1 + \varepsilon, \tag{6}$$

along with the three kinematic variables ω_k, are the four generalized strains.

The force and couple resultants acting on the side of a cross-section facing in the direction of increasing S are denoted by the vectors $\mathbf{N}$, $\mathbf{M}$, respectively. The distributed force $\mathbf{f}$ and the distributed couple $\mathbf{g}$ act along the rod center-line, where $\mathbf{f}$, $\mathbf{g}$ are referred to unit length in the undeformed rod. The differential equations of equilibrium can be written as:

$$\frac{\partial \mathbf{N}}{\partial S} + \mathbf{f} = 0, \qquad \frac{\partial \mathbf{M}}{\partial S} + (1+\varepsilon)\mathbf{a}_3 \times \mathbf{N} + \mathbf{g} = 0. \tag{7}$$

The components of $\mathbf{N}$, $\mathbf{M}$, $\mathbf{f}$, $\mathbf{g}$ are referred to current base vectors, i.e.

$$N_i = \mathbf{a}_i \cdot \mathbf{N}, \qquad M_i = \mathbf{a}_i \cdot \mathbf{M}, \qquad f_i = \mathbf{a}_i \cdot \mathbf{f}, \qquad g_i = \mathbf{a}_i \cdot \mathbf{g}. \tag{8}$$

The mechanical power P, per unit of length measured in the undeformed rod, is given by

$$P = N_3\dot{\varepsilon} + M_k\dot{\omega}_k. \tag{9}$$

Equations (1)–(9) hold for arbitrarily-large deformations. For small strains in a thin elastic rod with a circular cross-section of radius c, constitutive relations can be taken simply as

$$N_3 = EA\varepsilon, \qquad M_1 = EI\omega_1, \qquad M_2 = EI\omega_2, \qquad M_3 = GJ\omega_3, \tag{10}$$

where

$$A = \pi c^2, \qquad 2I = J = \pi c^4/2$$

and E is Young's modulus, G the shear modulus.

When the center-line of a circular rod forms a helix in the undeformed state, it is convenient to orient the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, at $t = 0$, to concide with the unit principal normal and binormal vectors, respectively, of the center-line. It then follows from the well-known Frenet formulae and eqn (3), evaluated at $t = 0$, that

$$\mathrm{K}_{ij} = \begin{bmatrix} 0 & \mathrm{T} & -\mathrm{K} \\ -\mathrm{T} & 0 & 0 \\ \mathrm{K} & 0 & 0 \end{bmatrix}, \tag{11}$$

where K, T are constants denoting the values of the curvature and torsion, respectively, of the rod center-line in the undeformed state. In the currently deformed rod, it follows from eqns (3), (4) and (11) that

$$\kappa_{ij} = \begin{bmatrix} 0 & (\mathrm{T}+\omega_3) & -(\mathrm{K}+\omega_2) \\ -(\mathrm{T}+\omega_3) & 0 & \omega_1 \\ (\mathrm{K}+\omega_2) & -\omega_1 & 0 \end{bmatrix}. \tag{12}$$

The Frenet formulae can be used again, along with eqns (3), (6) and (12), to obtain relations between current values of the curvature and torsion of the rod center-line, denoted respectively by κ, τ, and the generalized strains ε, ω_k. These relations, linearized in ε, ω_k, are:

$$\kappa = \mathrm{K} - \mathrm{K}\varepsilon + \omega_2, \qquad \tau = \mathrm{T} - \mathrm{T}\varepsilon + \omega_3 - \mathrm{K}^{-1}\frac{\partial \omega_1}{\partial S}. \tag{13}$$

Also, the unit principal normal vector $\mathbf{n}$ and unit binormal vector $\mathbf{b}$ along the center-line in the current deformed state of the rod are related to the current values of $\mathbf{a}_1$, $\mathbf{a}_2$ by

$$\mathbf{n} = \mathbf{a}_1 - \mathrm{K}^{-1}\omega_1\mathbf{a}_2, \qquad \mathbf{b} = \mathrm{K}^{-1}\omega_1\mathbf{a}_1 + \mathbf{a}_2. \tag{14}$$

The coefficient ω_1/K in eqn (14) measures the small angle in the normal plane between $\mathbf{a}_1$ and $\mathbf{n}$, and between $\mathbf{a}_2$ and $\mathbf{b}$, in the current configuration.

For use later, the component forms of eqns (5) and (7), for a helical rod, are now noted. Since the right-hand side of eqn (5) is expressed in terms of current base vectors, it is convenient to refer $\mathbf{\Omega}$ to current base vectors. Thus

$$\Omega_i = \mathbf{a}_i \cdot \mathbf{\Omega}. \tag{15}$$

Equations, linearized for small strains, which express eqn (5) in terms of components are obtained by neglecting product terms of the form $\Omega_j\omega_k$. It then follows from eqns (3), (5), (12) and (15) that

$$\frac{\partial \Omega_1}{\partial S} - \mathrm{T}\Omega_2 + \mathrm{K}\Omega_3 = \dot{\omega}_1. \tag{16}$$

$$\frac{\partial \Omega_2}{\partial S} + \mathrm{T}\Omega_1 = \dot{\omega}_2, \tag{17}$$

$$\frac{\partial \Omega_3}{\partial S} - \mathrm{K}\Omega_1 = \dot{\omega}_3. \tag{18}$$

Similarly, equations of equilibrium, linearized for small strains, are obtained from eqns (7) and (8) as:

$$\frac{\partial N_1}{\partial S} - \mathrm{T}N_2 + \mathrm{K}N_3 + f_1 = 0, \tag{19}$$

$$\frac{\partial N_2}{\partial S} + \mathrm{T}N_1 + f_2 = 0, \tag{20}$$

$$\frac{\partial N_3}{\partial S} - \mathrm{K}N_1 + f_3 = 0, \tag{21}$$

$$\frac{\partial M_1}{\partial S} - \mathrm{T}M_2 + \mathrm{K}M_3 - N_2 + g_1 = 0, \tag{22}$$

$$\frac{\partial M_2}{\partial S} + \mathrm{T}M_1 + N_1 + g_2 = 0, \tag{23}$$

$$\frac{\partial M_3}{\partial S} - \mathrm{K}M_1 + g_3 = 0. \tag{24}$$

While eqns (13)–(24) have been linearized for small strains, they hold in the presence of arbitrarily-large rotations, inasmuch as components of $\mathbf{\Omega}$, $\mathbf{N}$ and $\mathbf{M}$ are referred to current base vectors $\mathbf{a}_i$, and the unit principal normal and binormal vectors $\mathbf{n}$ and $\mathbf{b}$ refer to the center-line in the current deformed state of the rod.

The angle ω_1/K in eqns (13) and (14) can be identified with the angle f in Love (1944). Love's theory is for an inextensible rod. When terms in ε in eqn (13) are dropped, the strains ω_2 and ω_3 correspond essentially to the changes in curvature and twist in Love's theory. However, since Love's treatment does not include an expression for mechanical power corresponding to eqn (9), Love's angle f is not directly related to the component of bending moment along the principal normal, as M_1 is related to ω_1 by eqns (9) and (10). The equilibrium equations (19)–(24) do coincide with the corresponding linearized equilibrium conditions in Love's theory. On the other hand, Love's theory has no counterpart to eqns (16)–(18). Hence the significant difference between the present theory and Love's theory lies in the kinematic relations and the constitutive equations.

LINEARIZED EQUATIONS FOR DISPLACEMENTS

Multilayered cables are made up of layers of helical wires. In the undeformed reference state, these helixes are conveniently described in terms of cylindrical coordinates (ρ, ϕ, z) and the associated orthonormal base vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$, $\mathbf{e}_z$. The center-line of a wire, in a typical layer, lies on a cylindrical surface of radius R. The initial cylindrical coordinates of the material point S on the wire center-line can be written as

$$\rho = R, \qquad \phi = (S/R)\sin\alpha, \qquad z = S\cos\alpha, \tag{25}$$

where the lay angle α is positive in a right-handed helix, and negative in a left-handed helix. It is convenient to put

$$\Phi = (S/R)\sin\alpha, \qquad Z = S\cos\alpha. \tag{26}$$

Then (R, Φ, Z) are the initial coordinates of the material point S in the fixed spatial cylindrical coordinate system (ρ, ϕ, z). The curvature and torsion of the wire center-line in the undeformed reference state are given by

$$\mathrm{K} = R^{-1}\sin^2\alpha, \qquad \mathrm{T} = R^{-1}\sin\alpha\cos\alpha. \tag{27}$$

When changes in wire diameters are neglected, the current cylindrical coordinates of the material point S on the wire center-line in the deformed cable become

$$\rho = R, \qquad \phi = \Phi + \psi, \qquad z = Z + w, \tag{28}$$

where

$$\psi = \psi(S, t), \qquad w = w(S, t),$$

are the displacement components in cylindrical coordinates. The wire center-line in the deformed cable is defined as a space curve by eqn (28), and current values of the three invariants s, κ, τ of this space curve can be determined in terms of ψ, w. The results, linearized in ψ, w, ε, are, from Ramsey (1988),

$$\frac{\partial s}{\partial S} = 1 + R\sin\alpha\frac{\partial\psi}{\partial S} + \cos\alpha\frac{\partial w}{\partial S}, \tag{29}$$

$$\kappa = \mathrm{K} - 2\mathrm{K}\varepsilon + 2\sin\alpha\frac{\partial\psi}{\partial S}, \tag{30}$$

$$\tau = \mathrm{T} - 2\mathrm{T}\varepsilon + R^{-1}\sin\alpha\frac{\partial w}{\partial S} + R\csc\alpha\frac{\partial^3 w}{\partial S^3} + \cos\alpha\frac{\partial\psi}{\partial S} - R^2\cot\alpha\csc\alpha\frac{\partial^3\psi}{\partial S^3}. \tag{31}$$

The terms in ε on the right-hand sides of eqns (30) and (31) arise from the approximation

$$\frac{\partial()}{\partial s} = (1+\varepsilon)^{-1}\frac{\partial()}{\partial S} \approx (1-\varepsilon)\frac{\partial()}{\partial S}, \tag{32}$$

which follows from eqn (6). The four generalized strains ε, ω_k can now be related to ψ, w by using eqns (6) and (13), along with the results (29)–(31). Hence

$$\varepsilon = R\sin\alpha\frac{\partial\psi}{\partial S} + \cos\alpha\frac{\partial w}{\partial S}, \tag{33}$$

$$\omega_2 = -\mathrm{K}\varepsilon + 2\sin\alpha\frac{\partial\psi}{\partial S}, \tag{34}$$

$$\omega_3 - \mathrm{K}^{-1}\frac{\partial\omega_1}{\partial S} = -\mathrm{T}\varepsilon + R^{-1}\sin\alpha\frac{\partial w}{\partial S} + R\csc\alpha\frac{\partial^3 w}{\partial S^3} + \cos\alpha\frac{\partial\psi}{\partial S} - R^2\cot\alpha\csc\alpha\frac{\partial^3\psi}{\partial S^3}. \tag{35}$$

The kinematic relations (33)–(35) comprise a set of three equations which relate six unknowns, the four generalized strains ε, ω_k and the two generalized displacements ψ, w. To complete the set of equations for the problem, it is necessary to use the equilibrium conditions and constitutive relations. In the equilibrium equations, the components of distributed force and distributed couple f_k, g_k, which take account of any interwire forces and couples, constitute additional unknowns. A condition of rotational symmetry applies to the wires in each layer, i.e. every wire in the same layer is in the same state of loading and deformation. Consideration of this condition, along with the fact that interwire reactions must be equal and opposite, and the assumption of frictionless contacts between the wires all suggest that the only possible wire interactions consist of normal contact stresses between the wires which produce a radially-directed force component f_1. Accordingly,

$$f_2 = f_3 = g_1 = g_2 = g_3 = 0, \tag{36}$$

and there is no transfer of externally-applied load from one wire to another. Equation (36) leads to a consistent, well-posed set of equations for the problem. The constitutive relations (10) determine N_3, M_k in terms of ε, ω_k. Then the equilibrium equations (19)–(24) and the kinematic relations (33)–(35) form a set of nine linear equations for the nine unknowns ε, ω_k, ψ, w, N_1, N_2 and f_1. The three kinematic relations (16)–(18), which introduce the additional kinematic variables Ω_k, are used in formulating boundary conditions for the clamped-end condition.

The system of nine equations can be solved conveniently by treating the two generalized displacements ψ, w as the fundamental unknowns, and obtaining two simultaneous equations for determining them. Since there are no constitutive relations for the transverse shear forces N_1, N_2, it is convenient to eliminate them at the outset using the equilibrium equations. From eqns (21), (27) and (36), it follows that

$$N_1 = R\csc^2\alpha\,\frac{\partial N_3}{\partial S}. \tag{37}$$

Also, eqns (20), (27), (36) and (37) yield the result

$$\frac{\partial N_2}{\partial S} = -\cot\alpha\,\frac{\partial N_3}{\partial S}. \tag{38}$$

Equation (38) simply states, in differential form, the condition that the axial component of load carried by a wire is constant along the length of the wire, i.e. $\partial N_z/\partial S = 0$, where

$$N_z = \mathbf{e}_z\cdot\mathbf{N} = N_2\sin\alpha + N_3\cos\alpha. \tag{39}$$

In reducing the remaining equations from the original set to a set of two, it is convenient to introduce $W = W(S,t)$ as a dimensionless form of w, and to use Φ as a dimensionless form of the independent variable S. Thus,

$$w = R\csc\alpha\,W, \qquad \frac{\partial()}{\partial S} = R^{-1}\sin\alpha()', \qquad ()' = \frac{\partial()}{\partial\Phi}. \tag{40}$$

Then, from eqns (6), (29) and (40), it follows that

$$\varepsilon = \sin^2\alpha\psi' + \cos\alpha W'. \tag{41}$$

When eqns (40) and (41) are used with (34), an expression for ω_2 can be written as

$$R\omega_2 = \sin^2\alpha(1+\cos^2\alpha)\psi' - \sin^2\alpha\cos\alpha W'. \tag{42}$$

N_1 is determined using eqns (10), (37) and (41). With N_1 known, eqn (22) can be used to solve for ω_1. In view of eqns (10), (36), (40)–(42), it follows that

$$R\omega_1 = -\sin\alpha\tan\alpha[1+\cos^2\alpha+(4R^2/c^2)\csc^2\alpha]\psi'' - [-\sin^2\alpha+(4R^2/c^2)\csc^2\alpha]W''. \tag{43}$$

Now that both ε and ω_1 have been determined, ω_3 can be found from eqn (35). When eqn (40) is used as well, it is found that

$$R\omega_3 = -[\tan\alpha+\sin 2\alpha+(4R^2/c^2)\sec\alpha\csc\alpha]\psi''' + \sin\alpha\cos^3\alpha\psi' + [2\sin\alpha-(4R^2/c^2)\csc^3\alpha]W''' + \sin^3\alpha W'. \tag{44}$$

So far, eqns (22) and (24) have not been used. These two remaining equations determine two simultaneous equations for ψ, W. Equation (22) is differentiated with respect to S, and eqn (36) noted. Then $\partial M_3/\partial S$ is replaced by KM_1, noting eqns (24) and (36), and $\partial N_2/\partial S$ is replaced by $-\cot\alpha\ \partial N_3/\partial s$, from eqn (38). Finally, substitutions are made using eqns (10), (27), (40)–(43). The result is:

$$A_1\psi'''' - A_2\psi'' + B_1 W'''' - B_2 W'' = 0, \tag{45}$$

where

$$A_1 = \sin\alpha\tan\alpha[1+\cos^2\alpha+(4R^2/c^2)\csc^2\alpha],$$
$$A_2 = -\sin\alpha\tan\alpha(1+\cos^2\alpha)+(4R^2/c^2)\sec\alpha\cos 2\alpha,$$
$$B_1 = -\sin^2\alpha+(4R^2/c^2)\csc^2\alpha,$$
$$B_2 = \sin^2\alpha+(4R^2/c^2)\csc^2\alpha\cos 2\alpha.$$

In reducing eqn (24) to an equation in ψ, W, it is convenient to express Young's modulus E in terms of the shear modulus G and Poisson's ratio ν, i.e.

$$E = 2G(1+\nu). \tag{46}$$

When substitutions are made in eqn (24), noting (36) and using eqns (10), (27), (40), (43), (44) and (46), it follows that

$$C_1\psi'''' - C_2\psi'' + D_1 W'''' - D_2 W'' = 0, \tag{47}$$

where

$$C_1 = \sec\alpha[1+2\cos^2\alpha+(4R^2/c^2)\csc^2\alpha],$$
$$C_2 = \cos^3\alpha+(1+\nu)\sin\alpha\tan\alpha[1+\cos^2\alpha+(4R^2/c^2)\csc^2\alpha],$$
$$D_1 = -2+(4R^2/c^2)\csc^4\alpha,$$
$$D_2 = -\nu\sin^2\alpha+(1+\nu)(4R^2/c^2)\csc^2\alpha.$$

It can be noted that the coefficients in eqns (45) and (47) are even functions of the lay angle α, and hence these coefficients have the same values for both right-handed and left-handed helixes of the same pitch.

SOLUTION OF LINEARIZED EQUATIONS FOR DISPLACEMENTS

The general solution of eqns (45) and (47) includes the solution for uniform extension and twisting of the cable, which can be written in terms of the nominal cable extensional strain $\varepsilon_z = \varepsilon_z(t)$, and unit twist $\theta = \theta(t)$ as

$$w = \varepsilon_z Z, \qquad \psi = \theta Z. \tag{48}$$

Equation (48) implies that the displacement components w, ψ of material points on wire center-lines in the cable cross-section Z = constant are the same for all wires in the cable. This point will be examined further later.

A semi-infinite cable initially occupying the region $z \geqslant 0$ is now considered. A solution of eqns (45) and (47) appropriate for describing the localized effect of constraint due to the end connection has the form:

$$\begin{aligned} \psi &= \psi_1 e^{\beta_1 \Phi} + \psi_2 e^{\beta_2 \Phi}, \\ W &= k_1 \psi_1 e^{\beta_1 \Phi} + k_2 \psi_2 e^{\beta_2 \Phi}, \end{aligned} \tag{49}$$

where

$$\Phi = (Z/R)\tan\alpha \tag{50}$$

$$\left.\begin{matrix} \beta_1^2 \\ \beta_2^2 \end{matrix}\right\} = (B_0 \pm \sqrt{B_0^2 - 4A_0C_0})/(2A_0), \tag{51}$$

$$\begin{aligned} A_0 &= A_1D_1 - B_1C_1, \\ B_0 &= A_2D_1 + A_1D_2 - B_1C_2 - B_2C_1, \\ C_0 &= A_2D_2 - B_2C_2, \\ k_1 &= -(A_1\beta_1^2 - A_2)/(B_1\beta_1^2 - B_2), \\ k_2 &= -(A_1\beta_2^2 - A_2)/(B_1\beta_2^2 - B_2), \end{aligned}$$

and $\psi_1 = \psi_1(t)$, $\psi_2 = \psi_2(t)$ remain undetermined. Equation (50) follows from eqn (26) and the condition of rotational symmetry. Thus the solution expressed by eqn (49) holds for all wires in the same layer, but differs from layer to layer. The right-hand side of eqn (51) depends on the geometric parameters α and R/c, and Poisson's ratio ν. When representative numerical values for these quantities are used to evaluate the right-hand side of eqn (51), it appears always to be positive. Then, since $\Phi = (Z/R)\tan\alpha$, it is appropriate to take the negative square roots for β_1 and β_2 when $\alpha > 0$, and the positive square roots when $\alpha < 0$. Thus a solution exhibiting exponential decay with distance Z from the end connection can always be found.

BOUNDARY CONDITIONS AT A CLAMPED END

Boundary conditions for the clamped-end condition are now formulated using eqns (16)–(18). In view of eqns (27) and (40), eqns (17) and (18) can be rewritten as

$$\Omega_2' = -\cos\alpha\Omega_1 + R\csc\alpha\,\dot{\omega}_2, \qquad \Omega_3' = \sin\alpha\Omega_1 + R\csc\alpha\,\dot{\omega}_3. \tag{52}$$

Next, eqn (16) is differentiated with respect to Φ, using eqn (40), and then substitutions are made from eqns (27) and (52). Hence Ω_1 satisfies the equation

$$\Omega_1'' + \Omega_1 = R\csc\alpha\dot{\omega}_1' + R\cot\alpha\dot{\omega}_2 - R\dot{\omega}_3. \tag{53}$$

The right-hand side of eqn (53) can be expressed directly in terms of ψ, W by using eqns (42)–(44). The result is

$$\Omega_1'' + \Omega_1 = \sin\alpha[\cos\alpha(\dot{\psi}''' + \dot{\psi}') - (\dot{W}''' + \dot{W}')]. \tag{54}$$

It is useful to introduce components of $\mathbf{\Omega}$ referred to the orthonormal base vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$, $\mathbf{e}_z$ in cylindrical coordinates, i.e.

$$\Omega_\rho = \mathbf{e}_\rho \cdot \mathbf{\Omega}, \qquad \Omega_\phi = \mathbf{e}_\phi \cdot \mathbf{\Omega}, \qquad \Omega_z = \mathbf{e}_z \cdot \mathbf{\Omega}$$

where

$$\Omega_\rho = -\Omega_1 \tag{55}$$

$$\Omega_\phi = -\cos\alpha\Omega_2 + \sin\alpha\Omega_3, \tag{56}$$

$$\Omega_z = \sin\alpha\Omega_2 + \cos\alpha\Omega_3. \tag{57}$$

Equations (56) and (57) are differentiated with respect to Φ, and then substitutions are made from eqn (52). Thus

$$\Omega_\phi' = \Omega_1 - R\cot\alpha\dot{\omega}_2 + R\dot{\omega}_3, \tag{58}$$

$$\Omega_z' = R\dot{\omega}_2 + R\cot\alpha\dot{\omega}_3. \tag{59}$$

Equation (58) can be rewritten using eqn (53) to express Ω_1 in terms of Ω_1''. The resulting equation can be integrated once with respect to Φ. Hence

$$\Omega_\phi = -\Omega_1' + R\csc\alpha\dot{\omega}_1. \tag{60}$$

No constant of integration is included on the right-hand side of eqn (60) because just the particular solution of eqns (16)–(18) is of immediate interest. The homogeneous solution of eqns (16)–(18) describes rigid-body rotation.

During extension and twisting of the cable, the clamp or socket which forms the end connection is treated as a rigid body having an angular velocity Ω_0 about the z-axis and a velocity of translation v_0 along the z-axis. The wires comprising the cable are assumed to be rigidly embedded inside the end connection. Thus, at the face of the end connection, in the cable cross-section $Z = 0$, the velocity components $\dot{\psi}$, $\dot{w}$ of material points on wire center-lines must match the corresponding velocity components of the end connection. Hence

$$\dot{w} = v_0 \quad (Z = 0), \tag{61}$$

$$\dot{\psi} = \Omega_0 \quad (Z = 0). \tag{62}$$

Also, the angular velocity components Ω_ρ, Ω_ϕ, Ω_z of wire cross-section, evaluated at $Z = 0$, must match the angular velocity components of the end connection. Thus,

$$\Omega_\rho = \Omega_\phi = 0 \quad (Z = 0), \tag{63}$$

$$\Omega_z = \Omega_0 \quad (Z = 0). \tag{64}$$

In view of eqns (55) and (60), eqn (63) can be rewritten as

$$\Omega_1 = 0 \quad (Z = 0), \tag{65}$$

$$\Omega_1' = R \csc\alpha \dot{\omega}_1 \quad (Z = 0). \tag{66}$$

Equations (61)–(66) hold uniformly for all wires in the cable.

The boundary conditions, eqns (65) and (66), are now used to determine ψ_1, ψ_2 in the solution given by eqn (49). The solutions described by eqn (48) and eqn (49) are superposed, and introduced on the right-hand side of eqn (54). The particular solution of eqn (54) can be written as

$$\Omega_1 = \bar{\Omega}_1 + p_1\dot{\psi}_1 e^{\beta_1\Phi} + p_2\dot{\psi}_2 e^{\beta_2\Phi}, \tag{67}$$

where

$$\bar{\Omega}_1 = R\dot{\theta}\cos^2\alpha - \dot{\varepsilon}_z \sin\alpha\cos\alpha, \quad p_1 = \beta_1 \sin\alpha(\cos\alpha - k_1), \quad p_2 = \beta_2 \sin\alpha(\cos\alpha - k_2). \tag{68}$$

Next, the right-hand side of eqn (67) is substituted in the boundary conditions, eqns (65) and (66). The two equations which result determine $\dot{\psi}_1$, $\dot{\psi}_2$ in terms of $\bar{\Omega}_1$:

$$p_1\dot{\psi}_1 + p_2\dot{\psi}_2 = -\bar{\Omega}_1, \tag{69}$$

$$[p_1\beta_1 + \csc\alpha(A_1 + B_1k_1)\beta_1^2]\dot{\psi}_1 + [p_2\beta_2 + \csc\alpha(A_1 + B_1k_2)\beta_2^2]\dot{\psi}_2 = 0. \tag{70}$$

Equations (40), (43) and (49) have been used in obtaining eqn (70). The solution of eqns (69) and (70) can be written as:

$$\dot{\psi}_1 = q_1\bar{\Omega}_1, \qquad \dot{\psi}_2 = q_2\bar{\Omega}_1. \tag{71}$$

The coefficients q_1, q_2 depend on α, R, c and ν.

The boundary conditions expressed by eqns (65) and (66) govern bending and twisting of the individual constituent wires in the transition region. Two of the remaining boundary conditions, eqns (61) and (62), govern axial and circumferential sliding of one layer of wires with respect to adjoining layers. In order that the complete solution for the displacements, formed by superposing eqns (48) and (49), meet the boundary conditions at $Z = 0$ on $\dot{w}$ and $\dot{\psi}$, it is necessary to add terms which are independent of Z. These additional terms do not affect the generalized strains or the boundary conditions already satisfied, namely eqns (65) and (66). Thus, when eqn (40) is noted

$$\psi = \psi_0 + \theta Z - \psi_1(1 - e^{\beta_1\Phi}) - \psi_2(1 - e^{\beta_2\Phi}) \tag{72}$$

$$w = w_0 + \varepsilon_z Z - R\csc\alpha[k_1\psi_1(1 - e^{\beta_1\Phi}) + k_2\psi_2(1 - e^{\beta_2\Phi})] \tag{73}$$

where ψ_0, ω_0 depend only on t, and

$$\dot{\psi}_0 = \Omega_0, \qquad \dot{w}_0 = v_0.$$

The last unused boundary condition, eqn (64) for Ω_z, affects relative rotation of the constituent wires. Equation (59) for Ω_z' is expressed in terms of ψ, W by using eqns (42) and (44). The result is immediately integrable with respect to Φ, and the solution for Ω_z which satisfies eqn (64) can be written as

$$\Omega_z = \Omega_0 + \dot{\theta}Z - [1 - \cos\alpha(C_1 + k_1D_1)\beta_1^2]\dot{\psi}_1(1 - e^{\beta_1\Phi}) - [1 - \cos\alpha(C_1 + k_2D_1)\beta_2^2]\dot{\psi}_2(1 - e^{\beta_2\Phi}), \tag{74}$$

when eqns (40), (50), (72) and (73) are used.

As a final result, expressions for the generalized strains are noted. The total strains ε, ω_k are written as the sum of two parts

$$\varepsilon = \bar{\varepsilon}+\varepsilon^*, \qquad \omega_k = \bar{\omega}_k+\omega_k^*, \tag{75}$$

where

$$\bar{\varepsilon} = R\theta \sin\alpha\cos\alpha+\varepsilon_z \cos^2\alpha, \tag{76}$$

$$R\bar{\omega}_1 = 0, \tag{77}$$

$$R\bar{\omega}_2 = R\theta\sin\alpha\cos\alpha(1+\cos^2\alpha)-\varepsilon_z\sin^2\alpha\cos^2\alpha, \tag{78}$$

$$R\bar{\omega}_3 = R\theta\cos^4\alpha+\varepsilon_z\sin^3\alpha\cos\alpha, \tag{79}$$

and

$$\varepsilon^* = (\sin^2\alpha+k_1\cos\alpha)\beta_1\psi_1 e^{\beta_1\Phi}+(\sin^2\alpha+k_2\cos\alpha)\beta_2\psi_2 e^{\beta_2\Phi}, \tag{80}$$

$$R\omega_1^* = -(A_1+k_1B_1)\beta_1^2\psi_1 e^{\beta_1\Phi}-(A_1+k_2B_1)\beta_2^2\psi_2 e^{\beta_2\Phi}, \tag{81}$$

$$R\omega_2^* = \sin^2\alpha(1+\cos^2\alpha-k_1\cos\alpha)\beta_1\psi_1 e^{\beta_1\Phi}+\sin^2\alpha(1+\cos^2\alpha-k_2\cos\alpha)\beta_2\psi_2 e^{\beta_2\Phi}, \tag{82}$$

$$R\omega_3^* = \sin\alpha[\cos^2\alpha+k_1\sin^2\alpha-(C_1+k_1D_1)\beta_1^2]\beta_1\psi_1 e^{\beta_1\Phi} + \sin\alpha[\cos^2\alpha+k_2\sin^2\alpha-(C_1+k_2D_1)\beta_2^2]\beta_2\psi_2 e^{\beta_2\Phi}. \tag{83}$$

In eqns (75)–(83), $\bar{\varepsilon}$, $\bar{\omega}_k$ denote the uniform values of the strains that prevail as $Z\to\infty$, and ε^*, ω_k^* denote the localized superposed strains that result from constraint due to the end connection. The components $\bar{\varepsilon}$, $\bar{\omega}_k$ are obtained by substituting from eqn (48) in eqns (40)–(44), while ε^*, ω_k^* are determined using eqns (41)–(44) and (49).

In thin rods of circular cross-section, the three-dimensional stresses and strains can reasonably be assumed to vary linearly over the cross-section. Then the generalized strains ω_k can be converted to three-dimensional strains at the wire surface by multiplying by c, the wire radius, i.e. $\pm c\omega_1$, $\pm c\omega_2$ are the strains in the extreme fibers due to the bending moments M_1, M_2, while $c\omega_3$ is the shear strain at the wire surface due to the twisting moment M_3.

ILLUSTRATION

Equations (80)–(83) have been evaluated numerically for a four-layer electric power transmission cable designated Bersimis 42/7 ACSR conductor (Lanteigne, 1985). The cable consists of a straight central wire and four layers of helical wires. Two cases of loading were considered: extension without twisting ($\varepsilon_z \neq 0$, $\theta = 0$), and twisting without extension ($\varepsilon_z = 0$, $\theta \neq 0$). In both cases, in all layers, the maximum magnitudes of the strains ε^*, ω_k^* occur right at the face of the end connection ($Z = 0$). For the strains ε^*, ω_1^*, ω_2^*, the magnitudes diminish to less than 10% of their maximum values within 100 mm of the end, the overall cable diameter being 35 mm. The strain component ω_3^*, which measures twisting of the wires, is more persistent, its magnitude diminishing only to less than 20% of the maximum within 100 mm of the end, in the outermost layer. For the case of extension without twisting, the maximum magnitudes of the three-dimensional strains ε^*, $c\omega_k^*$ are less than 3% of ε_z, where ε_z is the extensional strain in the straight central wire.

For the case of twisting without extension, a convenient reference strain is the shear strain on the surface of the straight central wire. With c_0 ($= 1.27$ mm) denoting the radius of the cross-section of the straight central wire, this strain is $c_0\theta$. In this case, the maximum magnitudes of the bending strains $c\omega_1^*$, $c\omega_2^*$ in the wires in the two outer layers are as large as $c_0\theta$. The other maximum strain magnitudes are much smaller. In a practical situation of combined extension and twisting, the reference shear strain $c_0\theta$ due to twisting would be likely to be quite small compared to ε_z. Hence, constraint due to the end connection would increase the strains by only a few per cent over the uniform values which prevail well away from the end.

DISCUSSION

In view of the numerical results for the ACSR conductor, the increase in the stresses in the constituent wires due to constraint of the end connection is of little or no consequence. A much more significant effect of constraint would be the relative motion of wire surfaces which occurs at interwire contacts, and the resulting wear and fretting damage under fluctuating load. In the absence of constraint due to the end connection, $\psi_1 = \psi_2 = 0$, and eqns (72)–(74) reduce to

$$\dot{\psi} = \Omega_z = \Omega_0 + \dot{\theta} Z, \tag{84}$$

$$\dot{w} = v_0 + \dot{\varepsilon}_z Z. \tag{85}$$

As a consequence of eqns (84) and (85), $\dot{\psi}$, Ω_z and $\dot{w}$ have the same respective values for all wires in a cable cross-section; thus there is no sliding of one layer of wires, circumferentially or axially, with respect to adjoining layers, and there is no relative rotation of wire cross-sections in the plane of the cable cross-section. The relative motion of wire surfaces at the interwire contacts and the frictional forces developed, in the case of uniform extension and twisting of a cable, have been discussed in detail by Ramsey (1990). The terms in ψ_1, ψ_2 in eqns (72)–(74) vary from layer to layer, and hence end constraint causes sliding of one layer of wires with respect to the adjoining layers, and also relative rotation of the wires. In the absence of friction, this relative motion of wire surfaces at the interwire contacts extends along the entire length of the cable. Friction would have the effect of damping out this relative motion, and restricting it to the vicinity of the end. In a multilayered cable with friction, there would be some transfer of load from one layer of wires to another, near the end.

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